STABILITY OF A LAMINAR BOUNDARY LAYER OF A POWER-LAW NON-NEWTONIAN LIQUID

V. V. Zametalin

UDC 532.501.3

The stability of a laminar boundary layer of a power-law non-Newtonian fluid is studied. The validity of the Squire theorem on the possibility of reducing the flow stability problem for a power-law fluid relative to three-dimensional disturbances to a problem with two-dimensional disturbances is demonstrated. A numerical method of integrating the generalized Orr–Sommerfeld equation is constructed on the basis of previously proposed [1] transformations. Stability characteristics of the boundary layer on a longitudinally streamlined semiinfinite plate are considered.

In this work, we will study the stability of a laminar boundary layer of fluids with a power rheological law, for which the relation between the stress tensor τ_{ij} and strain rate tensor $\dot{e}_{ij}$ has the form

$$\tau_{ij} = -\delta_{ij}p + k\left|\frac{1}{2}\dot{e}_{ml}\dot{e}_{lm}\right|^{\frac{n-1}{2}}\dot{e}_{ij}, \tag{1}$$

where i, j = 1, 2, 3; k is the consistency index; n is the non-Newtonian index; p is pressure; and δ_{ij} is the Kronecker symbol. We will assume that media corresponding to $n > 1$ are dilating, while media with values $n < 1$ are pseudoplastic, the latter including, in particular, aqueous solutions of high polymers. The case $n = 1$ corresponds to a Newtonian fluid.

The flow stability of a power-law non-Newtonian fluid has been considered for a plane channel in [2]. The stability of the boundary layer of a power-law fluid was studied in [3, 4] using asymptotic methods. The critical Reynolds numbers for $0.2 \leq n \leq 2$ was estimated in [3] using an approximate formula derived there. Neutral stability curves for dilating fluids have been constructed in [4]. Some results [3, 4] are contradictory in the range of values $n > 1$. This leads to the necessity of a more careful determination of the stability characteristics, which can be attained by numerically integrating the stability equations. In the current work, the stability of the boundary layer of a power-law fluid is studied numerically on the basis of a previous method [1].

Well-known differential motion equations of a power-law non-Newtonian fluid [5] are obtained by substituting Eq. (1) in a stressed deformable continuum equation. Let us represent a nonsteady disturbed flow as the sum of two flows, namely, a steady main flow and small disturbing flow. As usual [6], we will assume that the main flow is laminar and that the components of disturbed motion can be represented in the form

$$\tilde{u} = u(y)e^{i(\alpha x+\beta z)-i\alpha ct};$$
$$\tilde{v} = v(y)e^{i(\alpha x+\beta z)-i\alpha ct};$$
$$\tilde{w} = w(y)e^{i(\alpha x+\beta z)-i\alpha ct};$$
$$\tilde{p} = p(y)e^{i(\alpha x+\beta z)-i\alpha ct},$$

where α and β are real values and $c = c_r + ic_i$ is complex.

The disturbing motion is assumed to be three-dimensional, since it has been proved [7] that the Squire theorem does not hold in the general case for non-Newtonian fluids.

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 101-106, January-February, 1976. Original article submitted November 29, 1974.

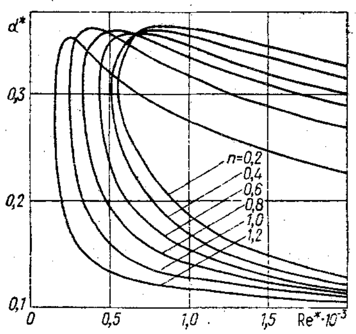

Fig. 1. The parameter n increases inversely to that indicated, i.e., the curve n = 1.2 corresponds to n = 0.2, etc.

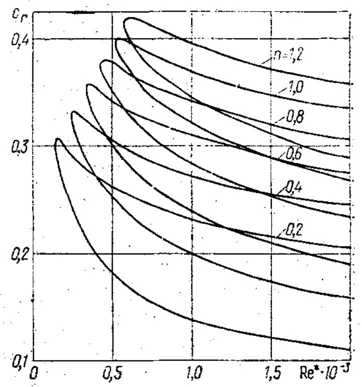

Fig. 2

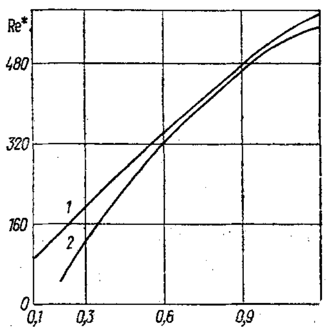

Fig. 3

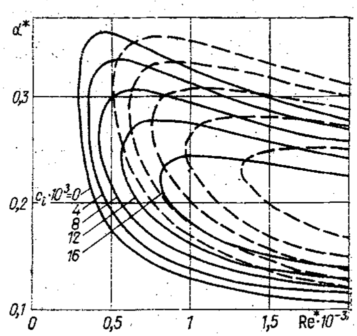

Fig. 4

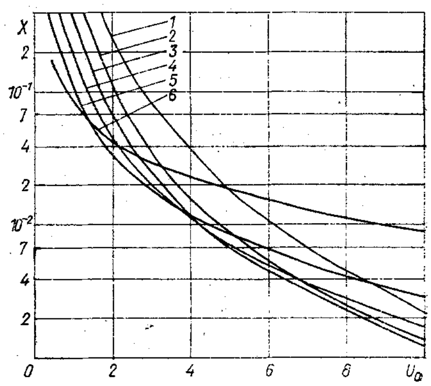

Fig. 5

We linearize the dimensionless differential equations for total flow, subtracting from them the main flow equations, obtaining

$$i\alpha \mathrm{Re}\,u(U-c)+vU'\mathrm{Re}+i\alpha \mathrm{Re}\,p=(U')^{n-1}[nu''-(\alpha^2+\beta^2)u+(n-1)i\alpha v']+(n-1)n(U')^{n-2}U''(u'+i\alpha v); \qquad (2)$$

$$i\alpha \mathrm{Re}\,v(U-c)+\mathrm{Re}\,p'=(U')^{n-1}[v''-(\alpha^2+\beta^2)v+i\alpha(n-1)(u'+i\alpha v)]+2(n-1)(U')^{n-2}U''v';$$

$$i\alpha \mathrm{Re}\,w(U-c)+i\beta \mathrm{Re}\,p=(U')^{n-1}[w''-(\alpha^2+\beta^2)w]+(n-1)(U')^{n-2}U''(w'+i\beta v);\quad i(\alpha u+\beta w)+v'=0,$$

where U(y) is main flow rate, $\mathrm{Re}=\rho U_{char}^{2-n}L_{char}^{n}/k$ is the generalized Reynolds number (ρ is fluid density). The primes denote differentiation with respect to the dimensionless tranverse coordinate y.

Thus, we have four equations to determine the four variables u, v, w, and p in the case of a three-dimensional disturbing motion.

Equations (2), after the transformations

$$\bar{\alpha}^2=\alpha^2+\beta^2,\quad \bar{p}\,\bar{\mathrm{Re}}=p\mathrm{Re},\quad \bar{\alpha}\bar{\mathrm{Re}}=\alpha\mathrm{Re},\quad \bar{v}=v,$$
$$\bar{c}=c,\quad \bar{\alpha}\bar{u}=\alpha u+\beta w,$$

take the form

$$i\overline{\alpha}\mathrm{Re}\overline{u}(U-\overline{c})+\overline{v}U'\mathrm{Re}+i\overline{\alpha}\mathrm{Re}\overline{p}=(U')^{n-1}[n\overline{u}''-\overline{\alpha}^2\overline{u}+(n-1)i\overline{\alpha}\overline{v}']+(n-1)n(U')^{n-2}U''(\overline{u}'+i\overline{\alpha}\overline{v}); \quad (3)$$

$$i\overline{\alpha}\mathrm{Re}\overline{v}(U-\overline{c})+\mathrm{Re}\overline{p}'=(U')^{n-1}[\overline{v}''-\overline{\alpha}^2\overline{v}+i\overline{\alpha}(n-1)(\overline{u}'+i\overline{\alpha}\overline{v})]+2(n-1)(U')^{n-2}U''\overline{v}';\ i\overline{\alpha}\overline{u}+\overline{v}'=0.$$

Equations (3) correspond to two-dimensional disturbing motion with Reynolds number Re less than Re in Eqs. (2). The Squire theorem for a power-law non-Newtonian fluid therefore holds.

In place of Eqs. (2) we then consider the two-dimensional analog of Eqs. (2), which is obtained if we set w = 0 and $\beta = 0$. We introduce a dimensionless stream function of the disturbing motion in the form

$$\psi=\varphi(y)e^{i\alpha(x-ct)}$$

obtaining, after some algebra, the generalized Orr–Sommerfeld equation for a power-law fluid,

$$(U-c)(\varphi''-\alpha^2\varphi)-U''\varphi=\frac{(U')^{n-3}}{i\alpha\,\mathrm{Re}}\{(U')^2n(\varphi^{\mathrm{IV}}-2\alpha^2\varphi'' \quad (4)$$
$$+\alpha^4\varphi)+(n-1)\{2nU'U''\varphi'''+[4\alpha^2(U')^2+nU'U'''+n(n-2)(U'')^2]\varphi''$$
$$+2(n-2)\alpha^2U'U''\varphi'+n\alpha^2[U'U'''+(n-2)(U'')^2]\varphi\}\}.$$

When n = 1, the equation turns into the ordinary Orr–Sommerfeld equation for a Newtonian fluid. The generalized Reynolds number for the case of a boundary layer is written in the form

$$\mathrm{Re}=\rho U_0^{2-n}\delta^n/k,$$

where U_0 is the free-stream flow rate and δ is layer thickness.

The boundary conditions have the form

$$\varphi(0)=\varphi'(0)=0, \quad (5)$$
$$\varphi(\infty)=\varphi'(\infty)=0.$$

Transformations [1] proposed for the ordinary Orr–Sommerfeld equation were used in solving the problem (4), (5) for the eigenvalues.

We define the function D_i (i = 1, 2, 3, 4, 5, 6) by the equations

$$D_1=\begin{vmatrix}\varphi_1 & \varphi_2\\ \varphi_1' & \varphi_2'\end{vmatrix},\quad D_2=\begin{vmatrix}\varphi_1' & \varphi_2'\\ \varphi_1'' & \varphi_2''\end{vmatrix},\quad D_3=\begin{vmatrix}\varphi_1'' & \varphi_2''\\ \varphi_1''' & \varphi_2'''\end{vmatrix},$$
$$D_4=\begin{vmatrix}\varphi_1 & \varphi_2\\ \varphi_1''' & \varphi_2'''\end{vmatrix},\quad D_5=\begin{vmatrix}\varphi_1' & \varphi_2'\\ \varphi_1''' & \varphi_2'''\end{vmatrix},\quad D_6=\begin{vmatrix}\varphi_1 & \varphi_2\\ \varphi_1'' & \varphi_2''\end{vmatrix},$$

where φ_1 and φ_2 are the two partial solutions of Eqs. (4) satisfying conditions (5). Then Eq. (4) can be reduced to a system of six ordinary differential equations:

$$D_1'=D_6\quad D_4'=D_5+AD_4+GD_1+BD_6,$$
$$D_2'=D_5,\quad D_5'=D_3+AD_5+BD_2-ED_1,$$
$$D_3'=AD_3-GD_2-ED_6,\qquad D_6'=D_2+D_4,$$

where

$$A=2(1-n)U''(U')^{-1};$$

$$B=(1-n)\left[\frac{4\alpha^2}{n}+(n-2)(U'')^2(U')^{-2}+U'''(U')^{-1}\right]+2\alpha^2+\frac{i\alpha}{n}\mathrm{Re}(U-c)(U')^{1-n};$$

$$G=\frac{2}{n}(1-n)(n-2)\alpha^2U''(U')^{-1};$$

$$E=(1-n)\alpha^2[U'''(U')^{-1}+(n-2)(U'')^2(U')^{-2}]-\alpha^4-\frac{i\alpha}{n}\mathrm{Re}(U')^{1-n}[(U-c)\alpha^2+U''].$$

We obtain the simple condition $D_1(0)=0$ to find the eigenvalues. We carry out the normalization $Z_i=D_i/D_6$ and eliminate Z_5 by integrating the system $Z_5=Z_1Z_3+Z_2Z_6$, finally being left with

$$Z_1'=1-Z_1(Z_2+Z_4), \quad (6)$$
$$Z_2'=Z_1Z_3-Z_2^2,$$
$$Z_3'=AZ_3-GZ_2-E-Z_3(Z_2+Z_4),$$
$$Z_4'=B+GZ_1+AZ_4+Z_1Z_3-Z_4^2.$$

TABLE 1

No. of curve from Fig. 5	c	n	k
1	0,6	0,506	0,736
2	0,3	0,546	0,307
3	0,2	0,579	0,173
4	0,1	0,666	0,051
5	0,05	0,789	0,010
6	0	1	0,001

Since the influence of viscosity is absent and, consequently, so are the non-Newtonian properties outside the boundary layer, the boundary conditions at infinity (5) can be carried over to the external edge of the boundary layer (y = 1) in the usual fashion [4, 6]. We will write the boundary conditions in the form

$$\varphi''(1)+(1+\alpha)\varphi'(1)+\alpha\varphi(1)=0;$$
$$\varphi'''(1)+\alpha\varphi''(1)=0$$

for convenience in calculating values of $Z_i(1)$. We set

$$\varphi_1 = 1;\ \varphi_1' = -\alpha;\ \varphi_1'' = \alpha^2;\ \varphi_1''' = -\alpha^3;$$
$$\varphi_2 = 0;\ \varphi_2' = 1;\ \varphi_2'' = -1-\alpha;\ \varphi_2''' = \alpha(1+\alpha)$$

when y = 1, obtaining the boundary conditions for the system (5),

$$Z_1 = -\frac{1}{1+\alpha};\ Z_2 = -\frac{\alpha}{1+\alpha};\ Z_3 = 0;\ Z_4 = -\alpha. \tag{7}$$

Thus, the problem has been reduced to a solution of the system (6) with the boundary conditions (7). The condition $Z_1(0) = 0$ can be made to hold by varying α, Re, and c and the eigenvalues are thereby found.

A self-consistent solution of the boundary-layer equations of a power-law fluid for the case of plane longitudinal streamline of a semiinfinite plate was used to determine the velocity profile. The boundary-value problem for the ordinary differential equation

$$|F''|^{n-1}F'''+FF''=0,$$
$$F(0)=0,\ F'(0)=0,\ F'(\infty)=1$$

was numerically solved using the method of group transformations set forth in [8]. The calculated velocities reasonably agree with those presented in [5].

Stability characteristics of the boundary layer of a power-law fluid for the range of values of the non-Newtonian factor $0.1 \le n \le 1,2$, i.e., basically for pseudoplastic fluids of most interest from the practical point of view, were calculated based on this technique. Neutral stability curves are depicted in Figs. 1 and 2 ($\alpha^* = \alpha\delta^*/\delta$; $Re^* = Re\delta^*/\delta$; where δ^* is displacement thickness).

The dependence of the generalized critical Reynolds number Re* on the parameter n is depicted in Fig. 3 (curve 1), in which the monotonically increasing nature of this function is maintained as we pass through $n \doteq 1$. Thus the results qualitatively agree with the previous [3] data obtained asymptotically (curve 2). Qualitatively satisfactory coincidence is observed when $0.6 \le n \le 1$. On the other hand, an extremely substantial divergence of the curves occurs in the region of low n. Curves of equally increasing disturbances calculated for the boundary layer of a power-law fluid when n = 0.5 (solid curves) and a Newtonian fluid (broken curves) for identical c_i are depicted in Fig. 4.

The coordinates of points at which flow loses stability in the boundary layer as a function of the free-stream flow rate U_0 for aqueous solutions of the high-molecular-weight polymer ET-597 were calculated based on our results, using previous [9] data. The family of curves constructed for the different concentrations (Fig. 5) allows the stability of the boundary layer of non-Newtonian power-law and Newtonian fluids to be graphically compared. Concentration C in percent, the non-Newtonian factor n, and consistency index k[Hc^n/m^2] for the corresponding curves in Fig. 5 are depicted in Table 1. The value of k is replaced by the viscosity of pure water at 20°C when C = 0 and n = 1. It is clear in Fig. 5 that, in spite of the general decreasing tendency for stability with increasing non-Newtonian properties, a region exists for low U_0 where the calculated points at which stability is lost for the polymer solutions is situated downstream from that of pure water. Our results lead us to conclude that the non-Newtonian viscosity of high-polymer solutions exerts a destabilizing influence in the case of flow into the boundary layer.

LITERATURE CITED

1. N. A. Zheltukhin, V. V. Zametalin, and N. M. Terekhova, "Hydrodynamical stability of boundary layers with mass exchange," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 3 (1974).
2. A. M. Makarov, L. K. Martinson, and K. B. Pavlov, "Stability of two-dimensional flow of a non-Newtonian fluid with power rheological law," Inzh.-Fiz. Zh., 16, No. 5 (1969).
3. V. V. Skripachev, "Stability of the boundary layer of a power-law non-Newtonian fluid," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 6 (1968).
4. V. B. Amfilokhiev, A. Ya. Voitkunskaya, and I. V. Ivanov, "Stability of laminar flow of a power-law non-Newtonian fluid in the boundary layer of a flat plate," in: Bionics [in Russian], No. 6, Naukova Dumka, Kiev (1972).
5. A. M. Shul'man and B. M. Berkovskii, The Boundary Layer of Non-Newtonian Fluids [in Russian], Nauka i Tekhnika, Minsk (1969).
6. Lin Chia-chiao, Theory of Hydrodynamical Stability [Russian translation], IL, Moscow (1958).
7. A. T. Listrov, "Stability of parallel flow of non-Newtonian media," Dokl. Akad. Nauk SSSR, 164, No. 5 (1965).
8. V. A. Bubnov, "Certain boundary-value problems of boundary-layer theory in power-law fluids," in: Heat and Mass Exchange [in Russian], Vol. 3, Nauka i Tekhnika, Minsk (1968).
9. R. Monti, "Heat transfer in drag-reducing solutions," in: Progress in Heat and Mass Transfer, Vol. 5, Pergamon, Oxford (1972).

CONVERGENT SHOCK WAVE IN AN IDEAL ELASTIC NONHOMOGENEOUS MEDIUM

I. V. Simonov

UDC 539.374

The boundary-value problem for symmetric focusing of a shock wave in a medium with variable density under a constant load (model of a porous body with variable initial velocity) is solved. The solution asymptotic is studied. Focusing in a homogeneous medium has been previously studied [1]. One inverse problem related to the choice of the optimal pressure conditions is examined. Constraints on the applicability of the model are touched on.

Suppose a uniform load $p_0(t)$ is applied to the surface of a sphere (cylinder, layer) whose initial density is a differentiable function of the radius [$\rho = \rho(r)$] at a moment of time $t=0$. We assume that the load instantaneously attains a finite value $p_0(t) > 0$ and does not increase any further (the physical meaning of this condition is that of an explosion on the surface); the medium is ideal (without tangential stresses). The density of the medium at any point ρ_1 is set equal to a constant ($0 < \rho < \rho_1$) and remains constant if the pressure at this point reaches values arbitrarily greater than zero. This highly simplified model approximately describes the behavior of a body with variable porosity and uniform skeleton at high loads.

A shock wave will propagate from the surface to the center. The focusing process for the shock wave in a homogeneous medium has been studied in [1]. The purpose of the current report is to investigate the influence of nonhomogeneity on the motion of the medium behind the front of a convergent shock wave. In particular, the variation in the degree of cumulation of a shock wave is of some interest. It may be expected that, as in the case of an ideal gas of variable density [2], the choise of $\rho(r)$ can either weaken or intensify accumulation.

The following motion and continuity equations hold within the region bounded by the moving surface $r = R_1(t)$ and the shock wave front $r = R(t)$:

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 107-111, January-February, 1976. Original article submitted January 30, 1975.